N94-10611

COHERENT STATES AND UNCERTAINTY RELATIONS FOR THE DAMPED HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

Kyu-Hwang Yeon
Department of Physics, Chungbuk National University, Cheong Ju, Chung Buk 360-763, Korea

Chung-In Um
Department of Physics, College of Science, Korea University, Seoul 136-701, Korea

Thomas F. George and Lakshmi N. Pandey
Departments of Chemistry and Physics
Washington State University, Pullman, Washington, 99164-1046, USA

Abstract

Starting with evaluations of propagator and wave function for the damped harmonic oscillator with time-dependent frequency, exact coherent states are constructed. These coherent states satisfy the properties which coherent states should generally have.

Since Schrödinger[1] constructed the coherent states for the harmonic oscillator, they have been widely used to describe many fields of physics[2,3,4]. Glauber[5] has used coherent states to discuss photon statistics of the radiation field, and Nieto and Simmons[6] have constructed coherent states for particles in various potentials. Hartley and Ray[7] have obtained exact coherent states for a time-dependent harmonic oscillator on the basis of Lewis and Riesenfeld[8]. Recently Yeon, Um and George obtained exact coherent states for a damped harmonic oscillator with constant frequency[9] and also the propagator, wave function, energy expectation values, uncertainty relations and coherent states for a quantum forced time-dependent harmonic oscillator[10].

In this paper we evaluate the wave function and uncertainty relations and construct exact coherent states for the damped harmonic oscillator with time-dependent frequency described by the modified Caldirola-Kanai Hamitonian through the path integral method,

$$H = f(t)[e^{-\gamma t}\frac{p^2}{2m} + e^{\gamma t}\frac{m}{2}(\omega^2 + \frac{\gamma^2}{4f(t)} - \frac{f(t)'\gamma}{2f(t)^3})x^2] , \tag{1}$$

where $f(t)$ is dimensionless time-dependent function and has the value $f(t)\mid_{t=0}=1$.

Very recently, we have obtained the propagators and wave functions for the damped driven harmonic oscillator with an external driving force $F(t)$[11], driven coupled harmonic oscillator[12], quantum oscillator chains[13] and a mode of the electromagnetic field in a resonator with time-dependent characteristics of the internal medium[14] by the path integral method. Through similar calculations in the above papers we may evaluate the propagator for the Hamiltonian of Eq.(1) :

$$\begin{aligned} K(x,t;x',t') &= [\frac{m\omega e^{\frac{\gamma}{2}(t+t')}}{2\pi i\hbar\sin(\omega\int_{t'}^{t}f(t)\,dt)}]^{1/2}\exp\{\frac{im\omega}{2\hbar}[\cot(\omega\int_{t'}^{t}f(t)\,dt)-\frac{\gamma}{2\omega f(t)}]e^{\gamma t}x^2 \\ &- \frac{2e^{\frac{\gamma}{2}(t+t')}xx'}{\sin(\omega\int_{t'}^{t}f(t)\,dt)}+[\cot(\omega\int_{t'}^{t}f(t)\,dt)+\frac{\gamma}{2\omega f(t)'}]e^{\gamma t'}x'^2\}\ . \end{aligned} \tag{2}$$

The solution of the Schrödinger equation is given as the path-dependent integral equation with propagator K,

$$\psi(x,t)=\int K(x,t;x',0)\psi(x',0)\,dx'\ , \tag{3}$$

which gives the wave function $\psi(x,t)$ at time t in terms of the wave function $\psi(x',0)$ at time $t=0$. At $t=0$ the Hamiltonian [Eq.(1)] reduces to the Hamiltonian of a simple harmonic oscillator, and the corresponding wave function becomes

$$\psi(x',0)=(\frac{\sqrt{m\omega_0/\hbar}}{2^n n!\sqrt{\pi}})^{1/2}H_n(\sqrt{\frac{m\omega_0}{\hbar}}x')e^{-\frac{m\omega_0}{2\hbar}x'^2}\ . \tag{4}$$

Substitution of Eqs.(2) and (4) into Eq.(3) yields the wave function

$$\begin{aligned} \psi(x,t) &= (\frac{\sqrt{m\omega_0/\hbar}}{2^n n!\sqrt{\pi}})^{1/2}\frac{e^{\frac{\gamma}{4}t}}{\xi}\exp\{-i(n+1/2)\cot[\omega/\omega_0\cot(\omega\int_{t'}^{t}f(t)\,dt)+\frac{\gamma}{2\omega_0}]\} \\ &\times\ e^{Ax^2}H_n(Dx)\ , \end{aligned} \tag{5}$$

where

$$\xi^2=\frac{\gamma^4}{16\omega^2\omega_0^2}\sin^2(\omega\int_0^t f(t)\,dt)+\frac{\gamma}{2\omega}\sin(2\omega\int_0^t f(t)\,dt)+1\ , \tag{6}$$

$$\begin{aligned} A &= -\frac{m\omega_0}{2\hbar}\frac{e^{\gamma t}}{\xi^2}+i\frac{m\omega}{2\hbar}\frac{e^{\gamma t}}{\xi^2}\{\xi^2[\cot(\omega\int_0^t f(t)\,dt)-\frac{\gamma}{2\omega f(t)}] \\ &- [\cot(\omega\int_0^t f(t)\,dt)+\frac{\gamma}{2\omega}]\}\ , \end{aligned} \tag{7}$$

$$D=\sqrt{\frac{m\omega_0}{\hbar}}\frac{e^{\frac{\gamma}{2}t}}{\xi}\ ,\quad Re\ A=-\frac{D^2}{2}\ ,\quad \omega^2=\omega_0^2-\frac{\gamma}{4}\ . \tag{8}$$

To evaluate the uncertainty values, we calculate the quantities

$$\begin{aligned} <x>_{mn} &= \int_{-\infty}^{\infty}\psi_m^*(x,t)x\psi_n(x,t)\,dx \\ &= \frac{\sqrt{n+1}}{\sqrt{2D}}e^{i\theta(t)}\delta_{m,n+1}+\frac{\sqrt{n}}{\sqrt{2D}}e^{-i\theta(t)}\delta_{m,n-1} \\ &= \mu\delta_{m,n+1}+\mu^*\delta_{m,n-1}\ , \end{aligned} \tag{9}$$

$$
\begin{aligned}
< m \mid p \mid n > &= \int_{-\infty}^{\infty} \psi_m^*(x,t) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi_n(x,t)\, dx \\
&= \sqrt{n+1}(-i\frac{\sqrt{2}A\hbar}{D})e^{i\theta(t)}\delta_{m,n+1} + \sqrt{n}(-i\frac{\sqrt{2}A\hbar}{D})^* e^{-i\theta(t)}\delta_{m,n-1} \\
&= \eta\, \delta_{m,n+1} + \eta^* \delta_{m,n-1} \ , \qquad (10)
\end{aligned}
$$

$$
< m \mid x^2 \mid n >= \sqrt{(n+2)(n+1)}\ \mu^2\ \delta_{m,n+2} + (2n+1)\mu\mu^*\delta_{m,n} + \sqrt{n(n-1)}\ \mu^{*\,2}\delta_{m,n-2} \ , \qquad (11)
$$

$$
< m \mid p^2 \mid n >= \sqrt{(n+2)(n+1)}\ \eta^2\delta_{m,n+2} + (2n+1)\eta\eta^*\delta_{m,n} + \sqrt{n(n-1)}\ \eta^{*\,2}\delta_{m,n-2} \ , \qquad (12)
$$

$$
\begin{aligned}
< m \mid \frac{1}{2}(xp+px) \mid n > &= \sqrt{(n+2)(n+1)}\ \mu\,\eta\ \delta_{m,n+2} + \hbar\frac{ImA}{D^2}(2n+1)\delta_{m,n} \\
&+ \sqrt{n(n-1)}\ \mu^*\eta^*\delta_{m,n-2} \ , \qquad (13)
\end{aligned}
$$

where

$$
\theta(t) = \cot^{-1}[\frac{\omega}{\omega_0}\cot(\omega\int_0^t f(t)\ dt) + \frac{\gamma}{2\omega_0}] \ , \qquad (14)
$$

$$
\mu(t) = \frac{e^{i\theta(t)}}{\sqrt{2}D} = \sqrt{\frac{\hbar}{2m\omega_0}}\ \xi e^{-\frac{1}{2}\gamma t}e^{i\theta(t)} \ , \qquad (15)
$$

$$
\begin{aligned}
\eta(t) &= -i\hbar\frac{\sqrt{2}A}{D}e^{i\theta(t)} \\
&= i\sqrt{\frac{m\omega_0\hbar}{2}}\ \frac{1}{\xi}\exp[\frac{1}{2}\gamma t\{1 - i\frac{\omega}{\omega_0}[\xi^2(\cot(\omega\int_0^t f(t)\ dt) - \frac{\gamma}{2\omega f(t)}) \\
&- (\cot(\omega\int_0^t f(t)\ dt) + \frac{\gamma}{2\omega})]\}] \\
&= \sqrt{\frac{m\omega_0\hbar}{2}}\frac{1}{\xi}e^{\frac{1}{2}\gamma t}\beta(t)e^{i[\cot^{-1}\sigma(t)+\theta(t)]} \ , \qquad (16)
\end{aligned}
$$

$$
\sigma(t) = \frac{\omega}{\omega_0}\{\xi^2[\cot(\omega\int_0^t f(t)\ dt) - \frac{\gamma}{2\omega f(t)}] - [\cot(\omega\int_0^t f(t)\ dt) + \frac{\gamma}{2\omega}]\} \ , \qquad (17)
$$

$$
\beta(t) = \sqrt{1+\sigma^2(t)} \ . \qquad (18)
$$

With the help of Eqs.(9)-(12), the uncertainty relations in the various states can be obtained as

$$
\begin{aligned}
[(\Delta x)^2(\Delta p)^2]^{1/2}_{n+2,n} &= [(< x^2 > - < x >^2)(< p^2 > - < p >^2)]^{1/2}_{n+2,n} \\
&= \sqrt{(n+2)(n+1)}\mid \mu \mid\mid \eta \mid \\
&= \frac{\hbar}{2}\sqrt{(n+2)(n+1)}\ \beta(t) \ , \qquad (19)
\end{aligned}
$$

$$
[(\Delta x)^2(\Delta p)^2]^{1/2}_{n+1,n} = \frac{\hbar}{2}(n+1)\beta(t) \ , \qquad (20)
$$

$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n,n} = \frac{\hbar}{2}(2n+1)\beta(t) \ . \tag{21}$$

Changing $(n+1)$ to n and $(n+2)$ to n in Eqs. (20) and (21), respectively, we can easily obtain the uncertainty in the $(n-1,n)$ state and $(n-2,n)$ state.

Now we return to the coherent states. Before we construct the annihilation operator a and creation operator $a^{\dagger}$, we will briefly discuss the properties of the coherent states. These states can be defined by the eigenstates of the nonhermitian operator a,

$$a \mid \alpha >= \alpha \mid \alpha > . \tag{22}$$

Using the completeness relation for the number representations, we expand $\mid \alpha >$ as

$$\begin{aligned} \mid \alpha > &= e^{-(1/2)|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > \\ &= e^{-(1/2)|\alpha|^2} e^{\alpha a^{\dagger}} \mid 0 > , \end{aligned} \tag{23}$$

where $\mid 0 >$ is the vacuum or ground state and is independent of n. The calculation of $< \beta \mid \alpha >$ in Eq. (23) gives

$$< \beta \mid \alpha >= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\beta^*} \tag{24}$$

Here, Eq. (24) has nonzero values for $\alpha \neq \beta$, and thus the states are not orthogonal, but when $\mid \alpha - \beta \mid^2 \longrightarrow \beta$ the states become orthogonal.

Since the eigenvalues α of the coherent states are complex numbers $u + iv$, the completeness relation of the coherent states is written as

$$\int \mid \alpha >< \alpha \mid \frac{d^2\alpha}{\pi} = 1 \ , \tag{25}$$

where 1 is the identity operator and $d^2\alpha$ is given by $d(Re\ u)d(Im\ v)$.

From Eqs. (9),(10),(15) and (16), we have the relation

$$\eta\mu^* - \eta^*\mu = i\hbar \ . \tag{26}$$

We can define the annihilation operator a and creation operator $a^{\dagger}$ for the damped harmonic oscillator with time dependent frequency as

$$\begin{aligned} a &= \frac{1}{i\hbar}(\eta x - \mu p) \ , \\ a^{\dagger} &= \frac{1}{i\hbar}(\mu^* p - \eta^* x) \ , \end{aligned} \tag{27}$$

where the expressions of x and p by a and $a^{\dagger}$ are

$$\begin{aligned} x &= \mu^* a + \mu a^{\dagger} \ , \\ p &= \eta^* a + \eta a \ . \end{aligned} \tag{28}$$

Since η is not equal to μ in Eqs. (15) and (16), we can easily confirm that a and $a\dagger$ are not Hermitian operators, but the following relations are preserved :

$$
\begin{aligned}
&[x,p] = i\hbar \ , \\
&[a,a^{\dagger}] = 1 \ . \qquad (29)
\end{aligned}
$$

Here, the operators a and $a^{\dagger}$ are different from $a_0^{\dagger}$ and a_0, i.e., creation and annihilation operators of the harmonic oscillator, and can be expressed as

$$
\begin{aligned}
a &= \lambda a_0 + \nu a_0^{\dagger} \ , \\
a^{\dagger} &= \mu^* a_0 + \lambda^* a_0^{\dagger} \ . \qquad (30)
\end{aligned}
$$

Therefore, the coherent states of the damped harmonic oscillator with time-dependent frequency are the squeezed states of the simple harmonic oscillator.

We can evaluate the transformation function $< x \mid \alpha >$ from the coherent states to the coordinate representation $\mid x >$. From Eqs. (22) and (27) we have

$$
[\eta x - \mu \frac{\hbar}{i}\frac{\partial}{\partial x}] < x \mid \alpha > = i\hbar\alpha < x \mid \alpha > \ . \qquad (31)
$$

Solving this equation, we obtain the coordinate representation

$$
< x \mid \alpha > = N \exp[\frac{1}{\mu}\alpha x - (2i\hbar\mu)^{-1}\eta\, x^2] \ . \qquad (32)
$$

Here, N is the integal constant constant. Choosing N to satisfy Eq. (25), we find the eigenvectors of the operator a given in the coordinate representation $\mid x >$ as

$$
< x \mid \alpha > = \frac{1}{(2\pi\mu\mu^*)^{1/4}} \exp[\frac{1}{2i\hbar}\frac{\eta}{\mu}x^2 + \frac{\alpha}{\mu}x - \frac{1}{2} \mid \alpha \mid^2 - \frac{1}{2}\frac{\mu^*}{\mu}\alpha^2] \ , \qquad (33)
$$

where

$$
\begin{aligned}
(2\pi\mu\mu^*)^{-1/4} &= (\frac{m\omega_0}{\pi\hbar})^{1/4}\xi^{1/2}e^{-\frac{\gamma}{4}t} \ , \\
\frac{i\eta}{2\hbar\mu} &= -\frac{m\omega_0}{2\hbar}\frac{1}{\xi^2}e^{\gamma t}[1 - i\sigma(t)] \ , \qquad (34) \\
\mu^*\mu &= e^{-2i\theta(t)} \ .
\end{aligned}
$$

Next, we prove that a coherent state represents a minimum uncertainty state. With the help of the relation between $a, a^{\dagger}, x$ and p, we evaluate the expectation values of x, p, x^2 and p^2 in state $\mid \alpha >$ as follows :

$$
\begin{aligned}
< x > &= < \alpha \mid \mu^* a + \mu a^{\dagger} \mid \alpha > = \mu^*\alpha + \mu\alpha^* \ , \\
< p > &= < \alpha \mid \eta^* a + \eta a^{\dagger} \mid \alpha > = \eta^*\alpha + \eta\alpha^* \ , \\
< x^2 > &= \mu^{*2}\alpha^2 + \mu\mu*(1 + 2\alpha\alpha^*) + \mu^2\alpha^{*2} \ , \qquad (35) \\
< p^2 > &= \eta^{*2}\alpha^2 + \eta\eta^*(1 + 2\alpha\alpha^*) + \eta^2\alpha^{*2} \ .
\end{aligned}
$$

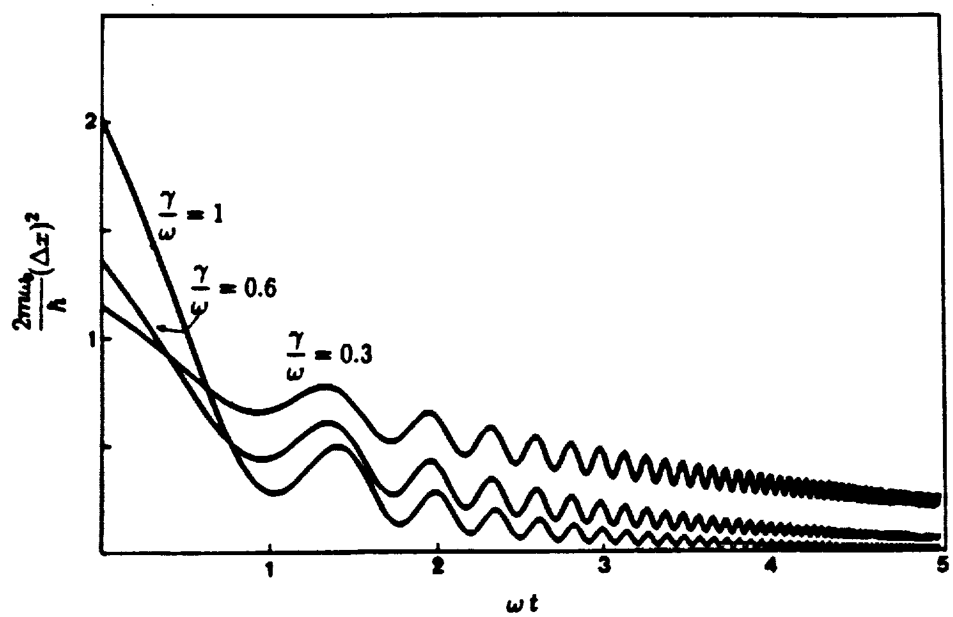

FIG. 1. $(\Delta x)^2$ for the (0,0) state as a function of ωt at various values of γ/ω with $\omega/\zeta = 1$.

From above expressions, we get

$$(\Delta x)^2 = \mu\mu^* = \frac{\hbar}{2m\omega_0}\xi^2 e^{-\gamma t} , \tag{36}$$

$$(\Delta p)^2 = \eta\eta^* = \frac{m\omega_0\hbar}{2}\xi^{-2}e^{\gamma t}\beta^2(t) , \tag{37}$$

and thus we finally obtain the uncertainty relation

$$(\Delta x)(\Delta p) = |\mu||\eta| = \frac{\hbar}{2}\beta(t) . \tag{38}$$

Equation (37) is the minimum uncertainty corresponding to Eq. (13) in the (0,0) state.

Taking $\gamma = 0$ and $f(t) = 1$, all the formulas we have derived are reduced to those of the simple harmonic oscillator. The propagator [Eq.(2)] and the wave function [Eq. (5)] do not have similar forms to those of Cheng[15] and others[16], but are of new form. We should point out that the same classical equation of motion can be obtained from many different action, and thus one may have many different propagators correspoding to the actions.

Figures 1, 2 and 3 illustrate the behaviors of $(\Delta x)^2, (\Delta p)^2$ and $\Delta p \cdot \Delta x$ as a function of ωt at various values of γ/ω and ω/ζ for $F(t) = e^{\zeta t}$ at $\gamma \neq 0$. When oscillation starts, $(\Delta x)^2$ and $(\Delta p)^2$ have the period Π, but their periods decrease rapidly with increasing time, and the amplitude of $(\Delta x)^2$ decreases exponentially, while that of $(\Delta p)^2$ increases exponentially. The uncertainty for the (0.0) state with period Π is reduced to that of the harmonic oscillator of 0° and 180°.

From all of the above results, we conclude that the coherent states for the damped harmonic

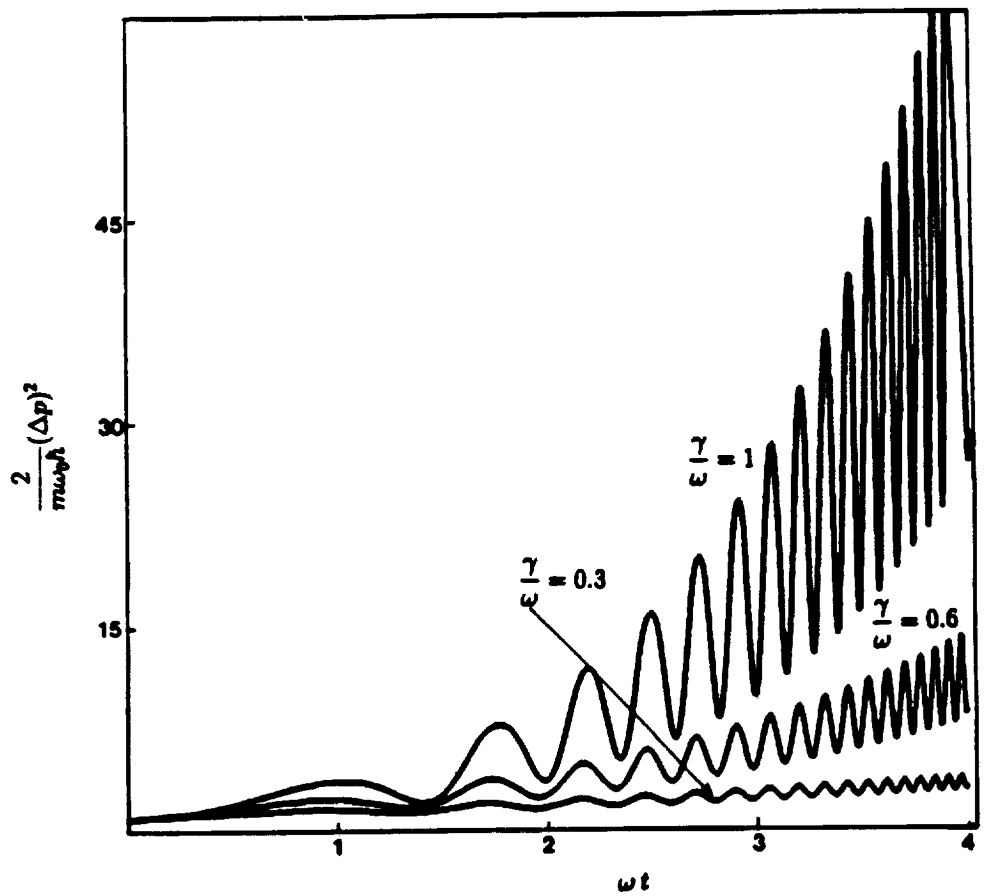

FIG. 2. $(\Delta p)^2$ for the (0,0) state a s a function of ωt at various values of γ/ω with $\omega/\zeta = 1$.

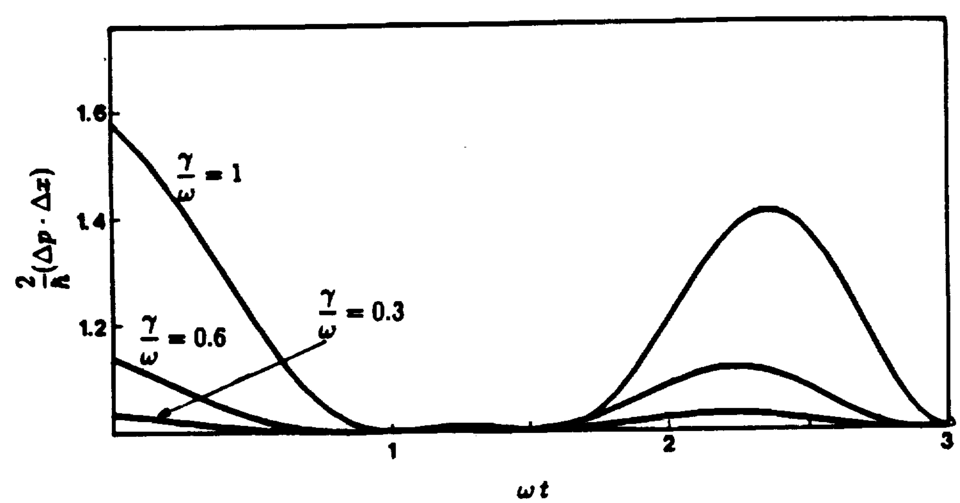

FIG. 3. $\Delta p \cdot \Delta x$ for the (0,0) state versus ωt at various values of γ/ω with $\omega/\zeta = 5$.

oscillator with the time-dependent frequency described by the modified Caldirola-Kanai Hamiltonian which we have constructed satisfy the renowned properties of coherent states.

Acknowledgments

This research was supported by the Center for thermal and Statistical Physics, KOSEF under Contract No. 91-08-00-05 and the National Science Foundation under Grant CHE-9196214.

References

[1] E. Schrödinger, Naturwissenshaften 14, 11 (1926).

[2] Z. E. Zimmerman and A. H. Silver, Phys. Rev. 167, 418 (1968).

[3] W. H. Louisell, *Quantum Statistical properties of Radiation* (Wiley, New York, 1973).

[4] J. C. Botke, D. J. Scalapino and R. L. Sugar, Phys. Rev. D 9 813 (1974).

[5] R. J. Glauber, Phys. Rev. 131, 2766 (1963) ; ibid. 130, 2529 (1963).

[6] M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D 20, 1321 (1979), ibid. 20, 1342 (1979) ; ibid. 20, 1342 (1979).

[7] J. G. Hartley and J. R. Ray, Phys. Rev. D 25, 382 (1982).

[8] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. 10, 1458 (1969).

[9] K. H. Yeon, C. I. Um and T. F. George, Phys. Rev. A 36, 5287 (1987)

[10] K. H. Yeon, C. I. Um and T. F. George, in *Workshop on Squeezed States and Uncertainty Relations*, edited by D. Han, Y. S. Kim and W. W. Zachary, NASA Conference Publication 3135, 347 (1991).

[11] C. I. Um, K. H. Yeon and W. H. Kahng, J. Phys. A : Math. Gen. 20, 611 (1987)

[12] K. H. Yeon, C. I. Um, W. H. Kahng and T. F. George, Phys. Rev. A 38, 6224 (1988).

[13] V. V. Dodonov, T. F. George, O. V. Man'ko, C. I. Um and K. H. Yeon, J. Sov. Laser Research 12, 385 (1991).

[14] V. V. Dodonov, T. F. George, O. V. Man'ko, C. I. Um and K. H. Yeon, J. Sov. Laser Research 13(N.4) in press (1992).

[15] B. K. Cheng, J. Phys. A 17, 2475 (1985).

[16] D. C. Khandekar and S. V. Lawanda, J. Math. Phys. 16, 384 (1975).